Stat Learn & Graph Models

HW 5

Due date: None

Problem 1 Identifying Markov Chains

A die is rolled repeatedly. Determine which of the following processes are a Markov chain (not necessarily finite-state). For Markov chains, find the transition probabilities.

Up to and including the nth roll:

a) Y_n: the largest number observed,

b) S_n: the sum of the values that have shown, and

c) F_n: the number of fives observed.

Solution

a) Y_n: This is a Markov chain since if we know what the maximum was in the previous step, then the values before that are irrelevant. Transition probabilities are given by

$$P_{ij} = \begin{cases} 0 & j < i \\ i/6 & j = i \\ 1/6 & j > i \end{cases}$$

for $1 \leq i, j \leq 6$.

b) S_n: This is a Markov chain with transition probabilities given by

$$P_{ij} = \begin{cases} 0 & j \leq i \\ 1/6 & i < j \leq i+6 \\ 0 & j > i+6 \end{cases}$$

for all positive integers i, j. Note that this is a chain with an infinite (but countable) number of states.

c) F_n: This is a Markov process with transition probabilities given by

$$P_{ij} = \begin{cases} 0 & j < i \\ 5/6 & j = i \\ 1/6 & j = i+1 \\ 0 & j > i+1 \end{cases}$$

Problem 2 Global balance property

Consider a Markov chain with the transition matrix

$$P = \begin{pmatrix} 1-p & p & 0 \\ p & 1-2p & p \\ 0 & p & 1-p \end{pmatrix}$$

where $p > 0$. Argue that the stationary distribution exists and is unique. Determine the stationary distribution.

Solution

We can go in two steps from every state to every other state. So the chain is regular. Thus the stationary distribution exists and is unique. The easiest way to find it is to use the global balance property. Let $\pi = (\pi_1, \pi_2, \pi_3)$ be the stationary distribution. By symmetry $\pi_1 = \pi_3$ and writing the GBP for state 1, we get $\pi_1 p = \pi_2 p$, which implies that $\pi_1 = \pi_2$, so $\pi = (1/3, 1/3, 1/3)$.

Problem 3 Kolmogorov Criteria for Reversibility

For a regular Markov chain with transition probabilities P_{ij}, show that if it is time-reversible then for any sequence $i_1, \ldots, i_m$ of states we have

$$P_{i_1 i_2} P_{i_2 i_3} \cdots P_{i_{m-1} i_m} P_{i_m i_1} = P_{i_1 i_m} P_{i_m i_{m-1}} \cdots P_{i_3 i_2} P_{i_2 i_1}.$$

Note: the converse also holds.

Solution

For regular Markov chains, time-reversibility is equivalent to the detailed balance property (DBP), which imples that $\pi_i P_{ij} = \pi_j P_{ji}$. Assume DBP. We have

$$\begin{aligned}
\pi_{i_1} P_{i_1 i_2} P_{i_2 i_3} \cdots P_{i_{m-1} i_m} P_{i_m i_1} &= P_{i_2 i_1} \pi_{i_2} P_{i_2 i_3} \cdots P_{i_{m-1} m} P_{i_m i_1} \\
&= P_{i_2 i_1} P_{i_3 i_2} \pi_{i_3} \cdots P_{i_{m-1} i_m} P_{i_m i_1} \\
&= \cdots = \\
&= P_{i_2 i_1} P_{i_3 i_2} \cdots \pi_{i_{m-1}} P_{i_{m-1} i_m} P_{i_m i_1} \\
&= P_{i_2 i_1} P_{i_3 i_2} \cdots P_{i_m i_{m-1}} \pi_{i_m} P_{i_m i_1} \\
&= P_{i_2 i_1} P_{i_3 i_2} \cdots P_{i_m i_{m-1}} P_{i_1 i_m} \pi_{i_1},
\end{aligned}$$

which is equivalent to the Kolmogorov criterion.

Problem 4 Poisson via Metropolis-Hastings

Describe the Metropolis-Hastings algorithm for sampling from a Poisson distribution with mean λ, that is, $p(x) \propto \frac{\lambda^x}{x!}$ (note that we do not need to know the normalizing factor). Use the following jump distribution:

$$J(x^*|x) = \begin{cases} 1/2 & x^* = x \pm 1 \\ 0 & \text{else} \end{cases}$$

Find the probability of acceptance (moving to a new value) explicitly. Use this to find the resulting Markov chain for $\lambda = 2$.

Solution

a) Let $x^0 = 1$.

b) For $t = 0, 1, 2, \ldots$

   i) Flip a fair coin and based on the result let $x^* = x^t + 1$ or $x^* = x^t - 1$.

   ii) We have
   $$r = \frac{p(x^*)/J(x^*|x^t)}{p(x^t)/J(x^t|x^*)} = \frac{p(x^*)}{p(x^t)}.$$

If $x^* = -1$, then $r = 0$. Otherwise, since $p(x) \propto \lambda^x/x!$,

$$r = \begin{cases} \frac{\lambda}{x+1} & x^* = x+1 \\ \frac{x}{\lambda} & x^* = x-1 \end{cases}$$

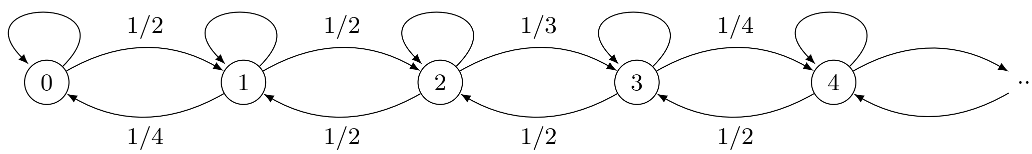
iii) Generate u randomly and uniformly from $[0, 1]$. If $u \leq r$, let $x^{t+1} = x^*$, otherwise, $x^{t+1} = x^t$.

Probability of acceptance is 0 for $x^* = -1$ and

$$\begin{cases} \min(1, \frac{\lambda}{x+1}) & x^* = x+1 \\ \min(1, \frac{x}{\lambda}) & x^* = x-1 \end{cases}$$

otherwise.

Noting that each of the options ($x^* = x + 1$ and $x^* = x - 1$) are chosen with probability 1/2, for $\lambda = 2$, we find the resulting Markov chain as given below. (The probabilities of the self loops can be computed from other probabilities and not given.)



It is easy to check that detailed balance in fact holds by writing the balance equations between x and $x + 1$ as $p(x)P(x \to x + 1) = p(x + 1)P(x + 1 \to x)$.