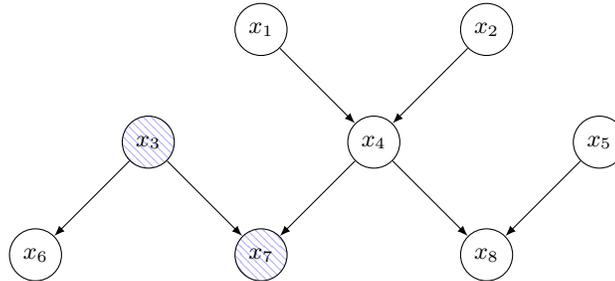


Stat Learn &amp; Graph Models

**HW 2**

Due date: 9/26/2017

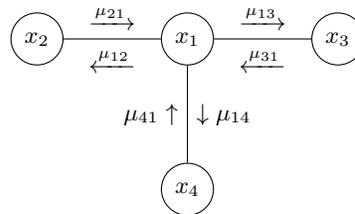
**Problem 1 Independence and d-separation**

Assume the distribution  $p(x_1^8)$  factorizes with respect to the graph above.

- (10 pts) Evaluate  $x_1 \perp\!\!\!\perp x_j | x_3, x_7$  for  $j \in \{2, 4, 5, 6, 8\}$ . For each case, find if this is true or false by identifying the path and determining if it is blocked or not. If the path is blocked indicate which node blocks it.
- (4 pts) Determine whether  $x_1 \perp\!\!\!\perp x_3 | x_3, x_7$  and  $x_1 \perp\!\!\!\perp x_7 | x_3, x_7$ .

**Problem 2 Elimination and sum-product**

In this problem, we will verify the results from sum-product by also finding them through elimination. In the following graph,  $\mu_{ij}$  denote messages passed in the sum-product algorithm.



Assume that

$$p(x_1^4) = \frac{1}{Z} \psi(x_1, x_2) \psi(x_1, x_3) \psi(x_1, x_4) \psi(x_1).$$

In your answers below, make sure you indicate the argument for any/all  $\mu$ .

- (12 pts) Determine all messages and put them in a correct order. Note that the correct order is not unique. For example, the first message may be:

$$\mu_{31}(x_1) = \sum_{x_3} \psi(x_1, x_3).$$

- (4 pts) Find the marginal of  $x_2$ , that is,  $p(x_2)$ , using the message(s) that it receives.

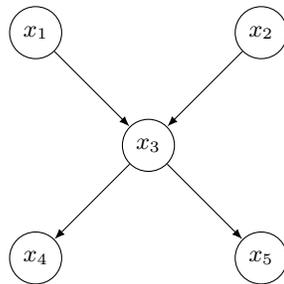
- c) (14 pts) Find  $p(x_2)$  through elimination, assuming the ordering  $x_3, x_4, x_1$  (first eliminate  $x_3, \dots$ ). Throughout the elimination process, whenever you come across a term that matches a message above, replace it with that message. For example, replace  $\sum_{x_3} \psi(x_1, x_3)$  with  $\mu_{31}(x_1)$ . Verify that we get the same expression as in sum-product. The elimination process starts as

$$p(x_2) = \frac{1}{Z} \sum_{x_1, x_4, x_3} \psi(x_1) \psi(x_1, x_2) \psi(x_1, x_3) \psi(x_1, x_4) = \dots$$

*Hint: In sum-product, we do not include  $Z$  in any message since we will normalize at the time of computing marginals. But in elimination, we normally carry  $Z$ . You must find a way to eliminate it at the end to show that both expressions are the same.*

### Problem 3 Converting graphical models

Consider the following Bayesian network,



with the corresponding distribution:

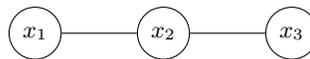
$$p(x_1^4) = p(x_1)p(x_2)p(x_3|x_1, x_2)p(x_4|x_3)p(x_5|x_3).$$

Find the equivalent:

- (8 pts) Markov random field and specify the potential functions.
- (8 pts) Factor graph and specify the factor at each factor node.

### Problem 4 Sum-product on chain with conditioning

(10 pts) Consider the following chain



with

$$p(x_1, x_2, x_3) = \frac{1}{Z} \psi(x_1, x_2) \psi(x_2, x_3).$$

Note that

$$\begin{aligned} p(x_3|\bar{x}_2) &= p(x_3|x_2 = \bar{x}_2) = \frac{p(\bar{x}_2, x_3)}{p(\bar{x}_2)} = \frac{\sum_{x_1} p(x_1, \bar{x}_2, x_3)}{\sum_{x_1, x_3} p(x_1, \bar{x}_2, x_3)} \\ &= \frac{\sum_{x_1} \psi(x_1, \bar{x}_2) \psi(\bar{x}_2, x_3)}{\sum_{x_1, x_3} \psi(x_1, \bar{x}_2) \psi(\bar{x}_2, x_3)} = \frac{\psi(\bar{x}_2, x_3)}{\sum_{x_3} \psi(\bar{x}_2, x_3)}. \end{aligned}$$

Prove the same result using sum-product message passing.

**Problem 5 Maximum Likelihood**

(10 pts) A coin with probability of heads equal to  $\theta$ ,  $P(H) = \theta$ , is flipped until heads shows. Let the number of times the coin is tossed be denoted by  $n$ . What is the ML estimate for  $\theta$  given  $n$ .